It should be obvious that the work required to add another order is significant. All of the mathematical operations in the equation must be redone with the new series.

Variational Process for Kepler's Equation

To derive the equations for an approximate analytical solution of Kepler's equation by the variational process, the solution (13) is expressed as

$$E_* = E + \delta E + \tfrac{1}{2!}\delta^2 E + \cdots \tag{21}$$

and it is desired to find the equations for the zeroth-order solution E, the first-order solution δE, and so on. Taking variations of Kepler's equation (14) leads to

$$\begin{aligned} E - e\sin E - M &= 0 \\ \delta E - \sin E\delta e + e\cos E\,\delta E &= 0 \\ \delta^2 E + 2\cos E\,\delta e\delta E - e\sin E\,\delta E^2 + e\cos E\,\delta^2 E &= 0 \\ &\vdots \end{aligned} \tag{22}$$

where $\delta(\delta e) = 0$ because δe is an independent variation. Then, evaluating the coefficients at $e = 0$ (E becomes the zeroth-order solution) gives the following:

$$\begin{aligned} E &= M \\ \delta E &= \sin E\,\delta e \\ \delta^2 E &= -2\cos E\,\delta e\delta E \\ &\vdots \end{aligned} \tag{23}$$

Once Eqs. (23) have been solved for $E, \delta E, \delta^2 E, \ldots$, the solution of Kepler's equation is given by Eq. (21).

That these equations are the same as Eqs. (20) can be shown by making the connections

$$\begin{aligned} \delta e = e, \qquad E = E_0, \qquad \delta E = E_1 e \\ \tfrac{1}{2!}\delta^2 E = E_2 e^2, \qquad \cdots \end{aligned} \tag{24}$$

The first equation comes from the fact that $\Delta e = \delta e$ because δe is an independent variation, that is, $\delta(\delta e) = 0$. Also, the total change in e is the value of e on the perturbed path minus the value of e on the nominal path, that is, $\Delta e = e - 0 = e$. The remaining equations come from comparing Eqs. (15) and (21).

To add another order is easy; only differentiation is needed. Admittedly, the complexity of each variation increases with order, but some of it goes away if terms can be combined.

Conclusions

Variational calculus has been developed for algebraic equations. It has been shown that Taylor series expansions can be made on a term by term basis by applying a differential (variational) process wherein variations of independent variations are zero and variations of dependent variations are not zero.

To establish their relative merits, the expansion process and the variational process have been applied to the algebraic perturbation problem in the form of Kepler's equation with small eccentricity. The variational process is clearly superior to the expansion process. In the expansion process, all of the mathematical operations must be carried out in terms of series, so that it is laborious to derive the equations for another-order solution. With the variational process, only differentiation is required, and the equation for another order can be obtained easily from the general form of the previous-order equation.

Usually, it is difficult to ensure that the various-order equations are correct regardless of the method being used. Having two ways to derive the equations helps alleviate this problem.

Although only a scalar equation is considered here, variational calculus can be applied to vector equations by employing indicial notation. If only the first-order correction to the zeroth-order solution is being sought, matrix notation can be employed.

Whereas the algebraic perturbation problem has been used to demonstrate the benefits of variational calculus, any problem involving a Taylor series expansion could have been used.

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Near Minimum-Time Trajectories for Solar Sails

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I. Introduction

SOLAR sailing has long been considered for a diverse range of future mission applications.[1] As with other forms of low-thrust propulsion, trajectory optimization has been a focus of development activities. In particular, minimum-time solar-sail trajectories have been obtained by several authors for a range of mission applications. Almost all of these studies have used the Pontryagin principle of the calculus of variations to obtain minimum-time trajectories by the classical, indirect method (see, for example, Ref. 2). The indirect approach provides a continuous time history for the required solar-sail steering angles. Only a few studies have used the competing direct approach, which recasts the task as a parameter optimization problem by discretizing the control variables. These studies have used many discrete segments for the sail steering angles to ensure a close approximation to the continuous steering angles provided by the indirect approach and hence a close approximation to the true minimum-time trajectory (see, for example, Ref. 3).

In this Note it is demonstrated that near minimum-time solar-sail trajectories can be obtained from the direct method using relatively few discrete segments. For trajectories involving transfers between near circular orbits, as few as three segments will yield a trajectory that satisfies the two-point boundary conditions of the trajectory while minimizing the transfer time to within a few percent of the absolute minimum transfer time. The motivation for this investigation arose from the observation that determination of minimum-time solar-sail trajectories is difficult because "the performance index is extremely insensitive to small variations around the optimal sail steering history."[4] Because the performance index is rather flat, it is expected that simple steering laws, which are operationally efficient to implement, will also be close to minimum time.

II. Fixed-Attitude Steering

To solve the minimum-time problem using direct methods, the sail steering angles are defined a priori in segments using a set of free parameters. Typically, the steering angles are parameterized in each segment as line elements or cubic splines to provide a continuous

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Table 1 Transfer time for Earth–Mars coplanar trajectory

N	T, days	Penalty, %
∞	408.2	0
20	408.8	0.15
10	409.0	0.20
5	410.0	0.44
4	418.0	2.40
3	438.0	7.30

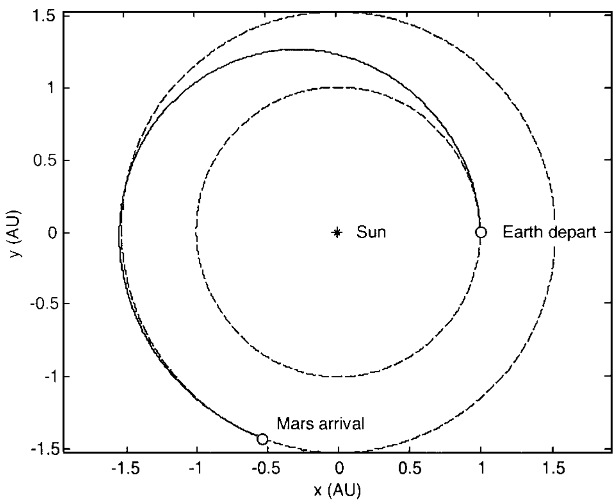

Fig. 1 Earth–Mars coplanar trajectory with N = 4.

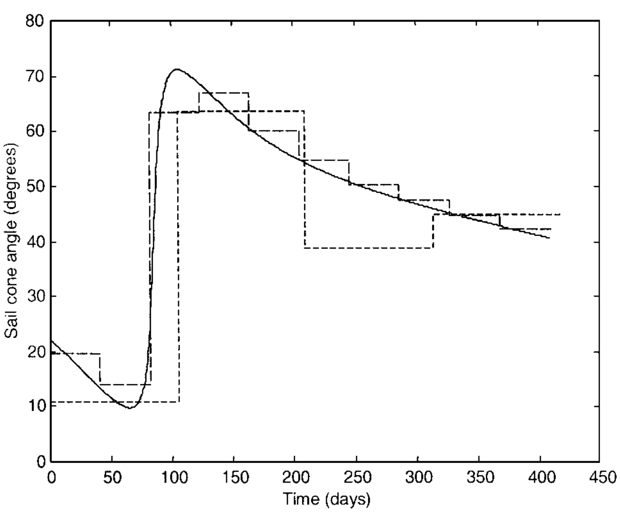

Fig. 2 Sail cone angle (- - - -, N = 4, – – –, N = 10, ——, $N \to \infty$).

time history for the resulting optimum steering angles. Many such segments are normally used to obtain a trajectory that is a close approximation to the true minimum-time trajectory. Rather than use interpolation methods, in this analysis the sail steering angles will be considered to be constant within each segment. Each segment is of equal length, so that for N segments the problem has N free parameters for a planar transfer and $2N$ parameters for a general three-dimensional transfer problem.

In principle this seems a disadvantageous approach because the steering law that is obtained will not be a good approximation to the continuous, true minimum-time steering law. However, operationally such a steering law is likely to be significantly easier to implement for a realistic solar sail. First, the sail can be stabilized about a fixed attitude relative to the sun line for long periods, rather than track a continuously changing attitude as required for true minimum-time trajectories. For some solar-sail configurations such a sun-fixed attitude could be maintained by passive means, thus further reducing the complexity of the problem. In addition, because of the small number of optimization parameters, onboard trajectory optimization can be contemplated. This can be the optimization of an entire trajectory for highly autonomous multiple objective missions, such as asteroid surveys, or reoptimization at the end of each segment to correct for uncertainties in the solar-sail force model and navigation errors. Although this approach will not produce a set of steering angles that closely matches the time history for a true minimum-time trajectory, the fact that the cost function of the problem is rather flat means that near minimum-time trajectories are still obtained. Any increase in transfer time is of little consequence for solar sails because no reaction mass is used. For other forms of low-thrust propulsion, true optimization is of course necessary to minimize launch mass.

III. Earth–Mars Trajectories

To illustrate the benefits of fixed-attitude steering, an Earth–Mars trajectory optimization problem will be considered. First, a coplanar transfer with an open final azimuthal angle will be considered and later a true three-dimensional rendezvous trajectory. For the coplanar problem the initial and final orbits are assumed to be circular with radii of 1 astronomical unit (AU) and 1.525 AU, whereas the rendezvous problem will use ephemeris data. In both cases the solar sail is assumed to have a characteristic acceleration of 1 mms.$^{-2}$ This is defined as the sail acceleration while facing the sun at 1 AU.

For the coplanar problem the true minimum time trajectory can be obtained using the Pontryagin principle and is found to be 408.2 days. This continuous case is listed in Table 1 as the number of steering angle segments $N \to \infty$. The problem can now be solved using a small number of fixed-sail cone angles, where the cone angle is defined as the angle between the sun line and the sail normal. This parameter optimization problem is now solved using sequential quadratic programming to minimize the transfer time while enforcing the trajectory boundary conditions as constraints.

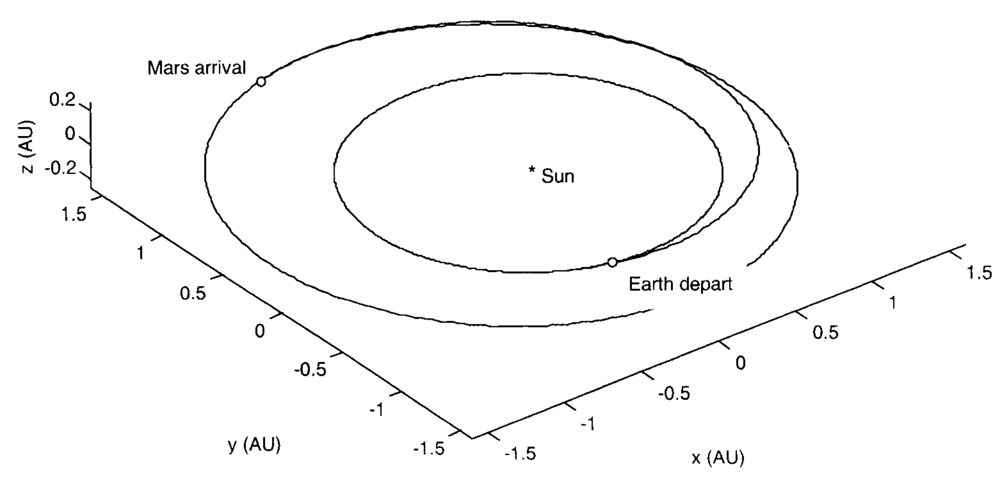

Fig. 3 Earth–Mars rendezvous trajectory with N = 5.

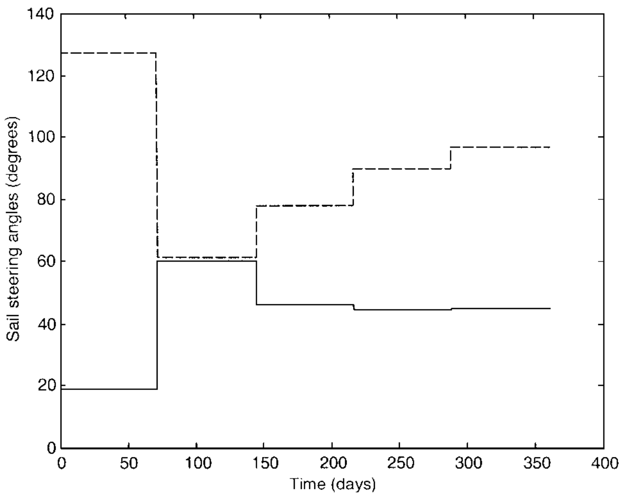

Fig. 4 Sail steering angles (——, cone angle, – – –, clock angle).

For a large number of trajectory segments, the transfer time is extremely close to the true minimum time, as expected. However, for a small number of segments there is only a modest penalty, as shown in Table 1. For example, using four fixed-sail attitudes during the entire trajectory the transfer time is increased by only 10 days. The trajectory obtained is shown in Fig. 1 while the steering angles are shown in Fig. 2. It can be seen that for a larger number of segments the steering law closely matches that obtained from the Pontryagin principle.

This approach can now be extended to rendezvous trajectories. For illustration a three-dimensional Earth–Mars transfer problem will be considered. A launch date of 13 May 1986 is chosen to allow comparison with the existing analysis of Sauer.[2] The minimum transfer time obtained from the Pontryagin principle is found to be 355.7 days. Again using sequential quadratic programming to minimize the transfer time while enforcing the boundary conditions as constraints, the transfer time using fixed steering angles is found to be 365.0 days for $N = 5$. For $N = 10$ the transfer time is reduced slightly to 362.0 days. The three-dimensional rendezvous trajectory to Mars capture obtained with $N = 5$ is shown in Fig. 3, whereas the sail steering angles are shown in Fig. 4. For three-dimensional transfers the sail clock angle is also required, defined to be the angle between the north ecliptic pole and a projection of the sail normal onto a plane normal to the sun line. Again, the ease of implementing only five fixed steering angles, rather than continuously tracking the true minimum-time steering angles, will more than offset the modest increase in transfer time.

IV. Conclusions

An investigation has been conducted into near minimum-time solar-sail trajectories. Using the observation that the cost function of the problem is rather flat means that near minimum-time trajectories can be obtained using only simple steering laws. Although these steering laws incur a modest penalty in transfer time, they are likely to provide significant operational benefits, particularly for future highly autonomous missions. Because solar sails do not require reaction mass, absolute trajectory optimization is of less importance than for other low-thrust spacecraft.

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Model Correction for Sampled-Data Models of Structures

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I. Introduction

THE modal analysis technique[1] has been extensively used throughout the literature to model dynamics of distributed systems such as flexible beams and plates, slewing beams, piezoelectric laminate beams, and acoustic ducts. Dynamics of such systems are described by partial differential equations (PDEs). In the modal analysis approach, the solution of these PDEs is allowed to consist of an infinite number of terms. Moreover, these terms are chosen to be orthogonal. Hence, modeling of a system based on a modal analysis approach can result in an infinite-dimensional model of that system.

In control design problems, one is often interested only in designing a controller for a particular frequency range. In these situations, it is common practice to remove the modes that correspond to frequencies that lie out of the bandwidth of interest and only keep the modes that directly contribute to the low-frequency dynamics of the system. Often two or more out-of-bandwidth modes may also be kept to improve the in-bandwidth model of the structure.

It is known that truncation has the potential to perturb the in-bandwidth zeros of the system. This problem is addressed in Ref. 2 and was recently revisited in Refs. 3 and 4. The mode acceleration method (see Ref. 2 page 350, and also Ref. 3) is concerned with capturing the effect of higher-frequency modes on the low-frequency dynamics of the system by adding a zero-frequency term to the truncated model to account for the compliance of the ignored modes. In Ref. 5, this problem is approached from an optimization perspective, where the DC content of the truncated model is modified to minimize the $\mathcal{H}_2$ norm of the error system that results from the truncation. In this paper, we concentrate on the sampled-data models of structures that are obtained by placing a sample and hold in the input of the system. We allow for a zero-frequency term to capture the effect of truncated modes and find this constant term such that the $\mathcal{H}_2$ norm of the resulting error system is minimized.

To this end, we point out that there are alternative methods to the modal analysis approach for modeling of distributed systems. However, the modal models have the interesting property that they describe spatial and temporal behavior of a system. Such models can then be used in designing spatial controllers as noted in Refs. 6 and 7.

II. Model Correction

Dynamics of a large number of distributed systems such as flexible beams, plates, and acoustic enclosures are governed by particular PDEs. Very often modal analysis is used, to solve these PDEs.[1] When modal analysis is used, a PDE can be shown to be equivalent to an infinite number of decoupled second-order, ordinary differential equations as

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = F_i u(t), \qquad i = 1, 2, \ldots \tag{1}$$

where $u(t)$ is the input of the system and q_i, $i = 1, 2, \ldots$, are the modal coordinates. Moreover, the input–output equation of the system in terms of a transfer function can be shown to be

$$G(s, r) = \sum_{i=1}^{\infty} \frac{\phi_i(r) F_i}{s^2 + \omega_i^2} \tag{2}$$

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